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An analysis of fixed point results in graphical fuzzy b-metric spaces with applications

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Abstract

Herein, a concept referred to as graphical fuzzy b-metric spaces is introduced, which generalizes the framework of fuzzy b-metric with the aid of graphical structures. Certain topological aspects corresponding to these spaces are explored, and related fixed point findings are established. These findings extend previously established results in the literature. Instances together with an application concerning equations of motion are included to showcase the feasibility of the established findings.

Keywords: Graphical fuzzy b-metric space; fuzzy graphical contraction; fixed point; equations of motion; directed graph.
2020 Mathematics Subject Classification: 54H25, 47H10, 05C20

1 Introduction

Fixed point theory remains an intensive research topic due to its wide range of applications across various domains. The groundbreaking Banach contraction principle [2] has laid down a solid foundation that underpins metric fixed point theory and has fostered subsequent investigations in various aspects, such as the development of novel contractive conditions and the study of generalized abstract spaces.

An important research direction concerns the connection between fixed point theory and fuzzy sets, as proposed by Zadeh [26]. Kramosil and Michalek [15] constructed a framework named fuzzy metric spaces. Later, George and Veeramani [12] proposed their revised version for ensuring consistency with the topological properties of classical metric spaces. As early as 1988, Grabiec [13] pioneered investigations into fixed point theory in these spaces, which has subsequently become a topic of ongoing research. Sedghi and Shobe [20] further advanced the framework by introducing fuzzy b-metric spaces, thereby establishing several contractive results relevant to that generalization.

Concurrently, graph theory continues to be a highly active research area in mathematics. Due to its graphical structure for representing objects and their relationships as vertices and edges, this theory has gained popularity and contributed significantly to various scientific disciplines such as chemistry, architecture, engineering, genetics,

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computer science, and mathematics [3, 8, 27]. Amid the current paradigm of big data, where the scale of generated information often goes beyond the scope of human analysis, graph learning and graph neural networks have received increasing attention in both academic and industrial fields. Moreover, the hypergraph, a generalization of the graph, has attracted considerable interest in the development of artificial intelligence and big data analysis, as it offers a framework for high-order correlations modeling (see [4, 7, 11]).

Over the past decade, interest in exploring the applicability of graph theory to the research on fixed point theory has increased notably. Jachymski [14] formulated a variant of the Banach contraction principle under a graph structure as a substitute for order structure on metric spaces. Subsequently, numerous researchers have continued to explore similar approaches. Notable studies on fixed point results utilizing graphs can refer to [1, 5, 10] and the references therein. In 2017, Shukla et al. [21] brought forth graphical metric spaces through graph-theoretic adaptation of classical metric paradigms. This adaptation involves reformulating the triangle inequality into a weaker version that applies exclusively to points lying on some paths in the space endowed with a graphical structure. Furthermore, a number of fixed point theorems were established with pertinent applications. Following this, de Hierro and Shahzad [9] addressed the findings of Shukla et al. [21] and discussed binary related distance spaces. From a mathematical perspective, the edge collection of a graphical structure defined over a given set forms a reflexive adjacency configuration. In metric fixed point theory, most conclusions build upon transitivity properties to prove fixed point findings subject to designated contractive criteria. In contrast, graphical metric spaces allow this condition to be omitted without compromising the attainment of fixed point. In 2018, Chuensupantharat et al. [6] advanced the original concept by introducing graphical b-metric framework and established corresponding contractive conclusions.

Recent advancements in graphical metric spaces and their relevant fixed point outcomes have motivated the exploration of graphical structure within fuzzy metric frameworks. In 2022, Saleem et al. [18] constructed a fuzzified adaptation of graphical metric spaces and attained several fixed point outcomes, along with an application. Serving as an original contribution to the field, their work raised some open problems, including the potential graphical extension of generalized fuzzy metric structures alongside the exploration of corresponding fixed point findings. Thereafter, Shukla et al. [22] revisited and addressed certain shortcomings in the work of [18]. These issues largely arise from the complexity of unifying fuzzy and graphical structures, where careful attention is necessary when defining the properties and establishing fixed points. In response, they introduced new fixed point results and explored various topological aspects in graphical fuzzy metric spaces.

Motivated by the works of [6], [18], and [22], the present work introduces the formalism of graphical fuzzy b-metric spaces as a novel advancement of extant research in this field. We explore certain topological properties of these newly defined spaces and establish several fixed point results. Illustrative examples, along with an application addressing equations of motion are included to substantiate our findings. The manuscript is divided as detailed below: Section 2 covers fundamental principles of graphical metric and fuzzy metric spaces. Section 3 introduces the construct for graphical fuzzy b-metric spaces and discusses their topological properties. Section 4 presents the core findings of the present work. Section 5 explores an application of our work in solving equations of motion. Finally, Section 6 concludes the paper with a discussion of open problems for potential future research.

2 Preliminaries

Current section covers fundamental notions and properties related to graphs, fuzzy metric spaces as well as graphical fuzzy metric spaces, providing an essential overview for understanding the content throughout the rest of the paper. Real numbers and positive integers are symbolized by $\mathbb{R}$ and $\mathbb{N}$, respectively. In order to preserve conceptual consistency found in the literature, we adopt the approach outlined in [21], [18], [14] and [12].

Suppose O is a nonempty set, while Δ represents the diagonal subset for $O \times O$. Let Γ be a directed graph without parallel edges, for which its vertex set $\Xi(\Gamma) = O$ and its edge set $\Sigma(\Gamma)$ includes all loops, i.e., $\Delta \subseteq \Sigma(\Gamma)$. In this manner, the set O can be regarded as associated with the graph $\Gamma = (\Xi(\Gamma), \Sigma(\Gamma))$, denoted simply by Γ. The converse graph Γ^{-1} is defined by

$$\Xi(\Gamma^{-1}) = \Xi(\Gamma) = O, \text{ and } \Sigma(\Gamma^{-1}) = \{(\rho, \varrho) \in O \times O : (\varrho, \rho) \in \Sigma(\Gamma)\}.$$

The undirected graph induced by graph Γ, which contains every symmetric edge, is denoted by $\tilde{\Gamma}$ where

$$\Xi(\tilde{\Gamma}) = \Xi(\Gamma), \text{ and } \Sigma(\tilde{\Gamma}) = \Sigma(\Gamma) \cup \Sigma(\Gamma^{-1}).$$

Consider vertices ρ, ϱ in Γ. Then, a path from ρ to ϱ of length $l \in \mathbb{N}$ in Γ comprises a sequence $\{x_i\}_{i=0}^{l}$ with $l+1$ vertices satisfying $x_0 = \rho$, $x_l = \varrho$ and $(x_{i-1}, x_i) \in \Sigma(\Gamma)$ for each $i = 1, 2, 3, \dots, l$. The graph Γ is called connected, if a path is present for each pair of vertices in Γ. Furthermore, for two vertices ρ and ϱ in a directed graph Γ, whenever a path from ρ to ϱ and from ϱ to ρ exists, then vertices ρ and ϱ are connected. The graph Γ is called weakly connected whenever its associated undirected graph $\tilde{\Gamma}$ is connected.

A relation P over O is defined by $(\rho P \varrho)_\Gamma$ precisely when a directed path exists from ρ to ϱ. Given a vertex $w \in O$, we write $w \in (\rho P \varrho)_\Gamma$ to indicate that w is located on a directed path from ρ to ϱ. For $l \in \mathbb{N}$, the notation $[\rho]_\Gamma^l$ is denoted as follows:

$$[\rho]_\Gamma^l = \{\varrho \in O : \text{a directed path connects } \rho \text{ to } \varrho \text{ of length } l\}.$$

Furthermore, a sequence $\{x_n\} \subseteq O$ is called Γ-termwise connected sequence whenever $(x_n P x_{n+1})_\Gamma$ holds for each $n \in \mathbb{N}$. A subgraph Γ' of Γ constitutes a graph with vertex and edge sets contained in those of Γ, that is, $\Xi(\Gamma') \subseteq \Xi(\Gamma)$ and $\Sigma(\Gamma') \subseteq \Sigma(\Gamma)$.

Hereafter, all graphs are assumed to be directed graphs and possess nonempty vertex and edge sets. Definitions regarding graphical metric spaces and graphical b-metric spaces follow.

Definition 1 ([21])**.** Suppose O is a nonempty set associated with a graph Γ and $d_\Gamma : O \times O \to \mathbb{R}$ be a function in which the conditions specified below hold:

(GM1) $d_\Gamma(x, y) \geq 0$ for all $x, y \in O$;

(GM2) $d_\Gamma(x, y) = 0$ if and only if $x = y$;

(GM3) $d_\Gamma(x, y) = d_\Gamma(y, x)$ for all $x, y \in O$;

(GM4) For $(xPw)_\Gamma$, $y \in (xPw)_\Gamma$ leads to $d_\Gamma(x, w) \leq d_\Gamma(x, y) + d_\Gamma(y, w)$ for all $w, x, y \in O$.

Consequently, d_Γ is known as graphical metric on O, whereby (O, d_Γ) is known as graphical metric space.

Definition 2 ([6])**.** Let O be a nonempty set associated with graph Γ with a real number $b \geq 1$. Suppose d_{Γ_b} : $O \times O \to \mathbb{R}$ is a function in which the conditions specified below hold:

(G_bM1) $d_{\Gamma_b}(x, y) \geq 0$ whenever $x, y \in O$;

(G_bM2) $d_{\Gamma_b}(x, y) = 0$ if and only if $x = y$;

(G_bM3) $d_{\Gamma_b}(x, y) = d_{\Gamma_b}(y, x)$, any $x, y \in O$;

(G_bM4) For $(xPy)_{\Gamma_b}$, $z \in (xPy)_{\Gamma_b}$ leads to $d_{\Gamma_b}(x, y) \leq b[d_{\Gamma_b}(x, z) + d_{\Gamma_b}(z, y)]$ for all $x, y, z \in O$.

Consequently, d_{Γ_b} is known as graphical b-metric on O, whereby (O, d_{Γ_b}) is known as graphical b-metric space.

Prior to presenting abstract fuzzy metrics as well as their generalizations, as given by the literature, we recall a fundamental concept essential to their formulations.

Definition 3 ([19])**.** Binary operation $* : [0,1] \times [0,1] \to [0,1]$ acts as continuous t-norm provided that subsequent properties hold:

(T1) $a * 1 = a$ whenever $a \in [0, 1]$;

(T2) $a * b = b * a$ for every $a, b \in [0, 1]$;

(T3) $c * (b * a) = (c * b) * a$ for every $a, b, c \in [0, 1]$;

(T4) $c \geq a$ and $d \geq b$ imply $a * b \leq c * d$ given $a, b, c, d \in [0, 1]$;

(T5) $*$ is continuous.

Example 1. Consider listed binary operations $*_m, *_p, *_L$ defined with $a, b \in [0, 1]$:

(i) $a *_m b = \min\{a, b\}$;

(ii) $a *_p b = ab$;

(iii) $a *_L b = \max\{0, a + b - 1\}$.

Thus, $*_m, *_p$ and $*_L$ are continuous t-norms.

Definition 4 ([12])**.** Let O be a nonempty set, $*$ be a continuous t-norm, and $M : O \times O \times (0, \infty) \to [0, 1]$ be a fuzzy set where the listed conditions are satisfied:

(FMS1) $M(x, y, t) > 0$;

(FMS2) $M(x, y, t) = 1$ if and only if $x = y$;

(FMS3) $M(x, y, t) = M(y, x, t)$;

(FMS4) $M(x, y, t + s) \geq M(x, z, t) * M(z, y, s)$;

(FMS5) $M(x, y, \cdot) : (0, \infty) \to [0, 1]$ is continuous;

considering $x, y, z \in O$ along with $t, s > 0$. Then, mapping M denotes fuzzy metric and the triple $(O, M, *)$ is termed a fuzzy metric space.

Definition 5 ([20])**.** Let O be a nonempty set, $*$ be a continuous t-norm, $b \geq 1$ belongs to $\mathbb{R}$ and $M : O \times O \times (0, \infty) \to [0, 1]$ be a fuzzy set where listed conditions are satisfied:

(F$_b$MS1) $M_b(x, y, t) > 0$;

(F$_b$MS2) $M_b(x, y, t) = 1$ if and only if $x = y$;

(F$_b$MS3) $M_b(x, y, t) = M(y, x, t)$;

(F$_b$MS4) $M_b(x, y, b(t + s)) \geq M(x, z, t) * M(z, y, s)$;

(F$_b$MS5) $M_b(x, y, \cdot) : (0, \infty) \to [0, 1]$ is continuous;

where $x, y, z \in O$ along with $t, s > 0$. Then, mapping M_b denotes fuzzy b-metric and the triple $(O, M_b, *)$ is termed a fuzzy b-metric space.

This section concludes with the fuzzy variant of graphical metric spaces, recently proposed and studied by Saleem et al. [18].

Definition 6 ([18])**.** Let O be a nonempty set associated with graph Γ, $*$ be a continuous t-norm, and $M_\Gamma : O \times O \times (0, \infty) \to [0, 1]$ be a fuzzy set in which conditions specified below hold:

(GFMS1) $M_\Gamma(x, y, t) > 0$;

(GFMS2) $M_\Gamma(x, y, t) = 1$ if and only if $x = y$;

(GFMS3) $M_\Gamma(x, y, t) = M_\Gamma(y, x, t)$;

(GFMS4) For $(xPy)_\Gamma$, $w \in (xPy)_\Gamma$ leads to $M_\Gamma(x, y, t + s) \geq M_\Gamma(x, w, t) * M_\Gamma(w, y, s)$;

(GFMS5) $M_\Gamma(x, y, \cdot) : (0, \infty) \to [0, 1]$ is continuous;

where $x, y, z \in O$ and $t, s > 0$. Consequently, the triple $(O, M_\Gamma, *)$ is known as graphical fuzzy metric space and M_Γ is known as graphical fuzzy metric on O.

3 Graphical Fuzzy b-Metric Spaces

This section proposes a novel extension of graphical fuzzy metric frameworks and explores certain properties of the newly defined spaces. These developments offer insight into fixed point theory from a graphical viewpoint, a topic of considerable current research. To begin with, the graph-theoretic conceptualization of fuzzy b-metric spaces is introduced.

Definition 7. Let O be a nonempty set associated with graph Γ, $*$ be a continuous t-norm, and $M_{\Gamma_b} : O \times O \times (0, \infty) \to [0, 1]$ be a fuzzy set. Suppose $b \geq 1$ is a real number, and the conditions specified below holds for M_{Γ_b}:

(GF$_b$MS1) $M_{\Gamma_b}(x, y, t) > 0$;

(GF$_b$MS2) $M_{\Gamma_b}(x, y, t) = 1$ if and only if $x = y$;

(GF$_b$MS3) $M_{\Gamma_b}(x, y, t) = M_{\Gamma_b}(y, x, t)$;

(GF$_b$MS4) For $(xPy)_\Gamma$, $z \in (xPy)_\Gamma$ implies $M_{\Gamma_b}(x, y, b(t + s)) \geq M_{\Gamma_b}(x, z, t) * M_{\Gamma_b}(z, y, s)$;

(GF$_b$MS5) $M_{\Gamma_b}(x, y, \cdot) : (0, \infty) \to [0, 1]$ is continuous;

where $x, y, z \in O$ and $t, s > 0$. Consequently, the triple $(O, M_{\Gamma_b}, *)$ is known as graphical fuzzy b-metric space and M_{Γ_b} is known as graphical fuzzy b-metric on O.

Example 2. Consider $O = \{1, 2, 3\}$ and the graph Γ in which $\Xi(\Gamma) = O$ and $\Sigma(\Gamma) = \Delta \cup \{(1, 2), (2, 3), (1, 3)\}$ (as shown in Figure 1). A fuzzy set $M_{\Gamma_b} : O \times O \times (0, \infty) \to [0, 1]$ is then defined by

$$M_{\Gamma_b}(x, y, t) = \frac{t}{t + (x - y)^2}, \text{ where } x, y \in O \text{ and } t > 0.$$

Consequently, $(O, M_{\Gamma_b}, *_p)$ is readily seen as a graphical fuzzy b-metric space, defined by $b = 2$.

Remark 1. As is evident, graphical fuzzy b-metrics are a generalization of graphical fuzzy metrics. Specifically, graphical fuzzy metric spaces represent a specific form of graphical fuzzy b-metric spaces with $b = 1$. Nevertheless, the instance below illustrates that the converse cannot be guaranteed in general.

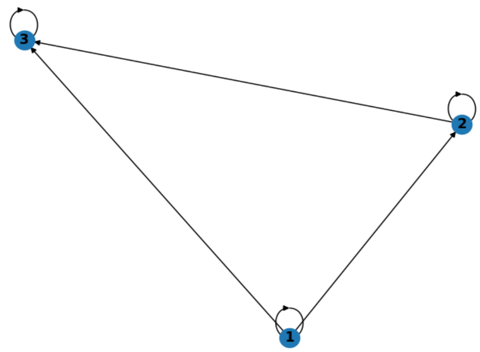

Figure 1: The graph structure corresponding to graphical fuzzy b-metric space.

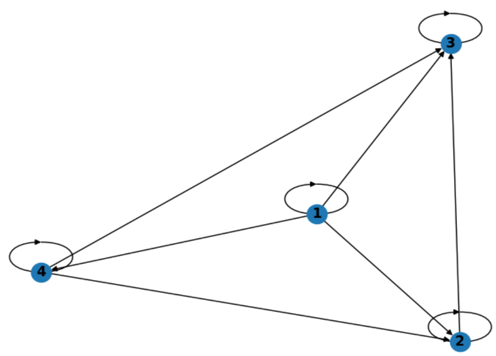

Figure 2: The graph structure corresponding to graphical fuzzy b-metric space.

Example 3. Consider $O = \{1, 2, 3, 4\}$ and the graph Γ defined by $\Xi(\Gamma) = O$ and $\Sigma(\Gamma) = O \times O$ (see Figure 2). A fuzzy set $M_{\Gamma_b} : O \times O \times (0, \infty) \to [0, 1]$ is then expressed as

$$M_{\Gamma_b}(x, y, t) = \exp\left\{-\frac{(x-y)^2}{t}\right\}, \text{ where } x, y \in O \text{ and } t > 0.$$

Consequently, $(O, M_{\Gamma_b}, *_m)$ forms a graphical fuzzy b-metric space, defined by $b = 4$.

We verify that it does not fulfill the conditions of a graphical fuzzy metric space. Consider case where $x = 1, y = 3, t = 5, s = 3$. It is clear that $(1P3)_\Gamma$ and $2 \in (1P3)_\Gamma$. The following computations are obtained:

$$\begin{aligned} M_{\Gamma_b}(1, 3, 8) &= 0.6065, \\ M_{\Gamma_b}(1, 2, 5) &= 0.8187, \\ M_{\Gamma_b}(2, 3, 3) &= 0.7165, \\ M_{\Gamma_b}(1, 2, 5) *_m M_{\Gamma_b}(2, 3, 3) &= \min\{0.8187, 0.7165\} = 0.7165. \end{aligned}$$

Consequently, condition (GFMS4) fails to hold, as $M_{\Gamma_b}(1, 3, 8) \ngeq M_{\Gamma_b}(1, 2, 5) *_m M_{\Gamma_b}(2, 3, 3)$. Hence, $(O, M_{\Gamma_b}, *_m)$ fails to qualify as graphical fuzzy metric space.

Remark 2. Each fuzzy b-metric space $(O, M_b, *)$ can be regarded as a graphical fuzzy b-metric space when equipped with Γ, where $\Xi(\Gamma) = O$ and $\Sigma(\Gamma) = O \times O$. However, a graphical fuzzy b-metric space need not be a fuzzy b-metric space. Subsequent instance demonstrates the preceding statement.

Example 4 ([22])**.** Consider $O = \left\{x_n : x_n = \frac{1}{n} \text{ for all } n \in \mathbb{N}\right\}$ and a graph Γ defined by $\Xi(\Gamma) = O$ and $\Sigma(\Gamma) = \Delta \cup \{(x_{n+1}, x_n) \in O \times O : n \in \mathbb{N}\}$ (see Figure 3). $M_{\Gamma_b} : O \times O \times (0, \infty) \to [0, 1]$ is expressed as

$$M_{\Gamma_b}(x, y, t) = \begin{cases} 1, & x = y; \\ xy, & \text{otherwise}; \end{cases}$$

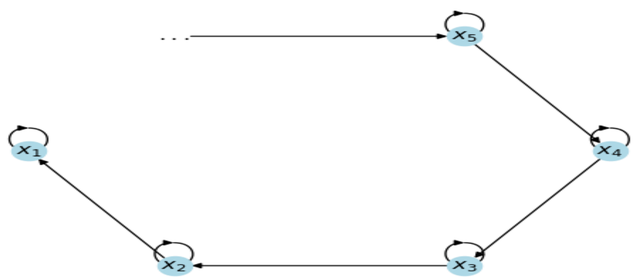

Figure 3: The graph structure corresponding to graphical fuzzy b-metric space.

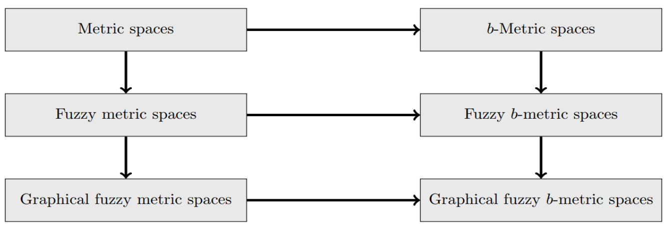

Figure 4: The arrows represent implications among the spaces, while the converse is not guaranteed.

where $x, y \in O$ and $t > 0$. Accordingly, $(O, M_{\Gamma_b}, *_m)$ is a graphical fuzzy b-metric space with any $b \geq 1$.

We verify that it does not fulfill the conditions of fuzzy b-metric space. Take $x, y, z \in O$ satisfying $x < y < z$. With this configuration, condition (F_bMS4) fails to hold, as $M_b(x, y, b(t+s)) \ngeq M(x, z, t) * M(z, y, s)$ for any $t, s > 0$. Hence, $(O, M_{\Gamma_b}, *_m)$ fails to qualify as a fuzzy b-metric space.

To enhance conceptual clarity, Figure 4 depicts the implication relationships between the abstract spaces, in accordance with the preceding discussion.

Definition 8. Consider $(O, M_{\Gamma_b}, *)$ to be a graphical fuzzy b-metric space. An open ball $B_{\Gamma_b}(x, r, t)$ centered at $x \in O$, with radius $r \in (0, 1)$ as well as $t > 0$, represents

$$B_{\Gamma_b}(x, r, t) = \{y \in O : (xPy)_\Gamma \text{ and } M_{\Gamma_b}(x, y, t) > 1 - r\}.$$

As $\Delta \subseteq \Sigma(\Gamma)$, it can be deduced that $x \in B_{\Gamma_b}(x, r, t)$. Hence, $B_{\Gamma_b}(x, r, t)$ is nonempty at every $x \in O, r \in (0, 1), t > 0$. A collection $\beta = \{B_{\Gamma_b}(x, r, t) : x \in O, r > 0 \text{ and } t > 0\}$ establishes neighborhood of topology τ_{Γ_b} over O arising from the graphical fuzzy b-metric M_{Γ_b}. Additionally, subset $U \subseteq O$ is considered open whenever for any $x \in U$, there exists $r \in (0, 1)$ in which $B_{\Gamma_b}(x, r, t)$ lies within U for all $t > 0$. Naturally, the complement set $O \setminus U$ is considered closed if U is open.

Lemma 1. *Let $(O, M_{\Gamma_b}, *)$ denote a graphical fuzzy b-metric space with topology τ_{Γ_b} arising from graphical fuzzy b-metric M_{Γ_b}. Then, τ_{Γ_b} is T_1 but not T_2(Hausdorff) in general.*

Proof. Assume $(O, M_{\Gamma_b}, *)$ is a graphical fuzzy b-metric space. It can be verified that for each $x \in O$, the singleton set $\{x\} \subset O$ is closed. In other words, the complement set $O \setminus \{x\}$ is open. Pick an arbitrary $y \in O \setminus \{x\}$. Thus, $x \neq y$, specifically, $1 > M_{\Gamma_b}(x, y, t) > 0$ for some $t > 0$. Fix $t^* > 0$ and consider $r = 1 - M_{\Gamma_b}(x, y, t^*) > 0$. It is evident that $x \notin B_{\Gamma_b}(y, r, t^*)$, otherwise $(yPx)_\Gamma$ and

$$\begin{aligned} M_{\Gamma_b}(x, y, t^*) = M_{\Gamma_b}(y, x, t^*) &> 1 - r \\ &= 1 - (1 - M_{\Gamma_b}(x, y, t^*)) \\ &= M_{\Gamma_b}(x, y, t^*), \end{aligned}$$

thereby producing a contradiction. Therefore, $B_{\Gamma_b}(y, r, t_0) \subset O \setminus \{x\}$, which validates $O \setminus \{x\}$ is open. □

Definition 9. Consider $(O, M_{\Gamma_b}, *)$, a graphical fuzzy b-metric space. A sequence $\{x_n\}$ in O is convergent to $x \in O$ if for each $r \in (0,1)$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ satisfying $M_{\Gamma_b}(x_n, x, t) > 1 - r$ for each $n \geq n_0$. Equivalently, the sequence $\{x_n\}$ in O is convergent to $x \in O$ if $\lim_{n\to\infty} M_{\Gamma_b}(x_n, x, t) = 1$ for any $t > 0$.

Definition 10. Consider $(O, M_{\Gamma_b}, *)$, a graphical fuzzy b-metric space. A sequence $\{x_n\}$ in O is Cauchy sequence if for every $r \in (0,1)$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ satisfying $M_{\Gamma_b}(x_n, x_m, t) > 1 - r$ for each $n, m \geq n_0$. Equivalently, the sequence $\{x_n\}$ is Cauchy sequence if $\lim_{n,m\to\infty} M_{\Gamma_b}(x_n, x_m, t) = 1$ for any $t > 0$.

Definition 11. A graphical fuzzy b-metric space $(O, M_{\Gamma_b}, *)$ is complete if all Cauchy sequences of O are convergent. Suppose Γ' is a different graph from Γ with $\Xi(\Gamma') = O$, then $(O, M_{\Gamma_b}, *)$ is considered Γ'-complete if each Γ'-termwise connected Cauchy sequence in O is convergent.

4 Main Results

This section aims to expound fixed point results concerning graphical fuzzy b-metric spaces. We commence by introducing several definitions that are essential for establishing the main results that follow.

Definition 12. Let $(O, M_{\Gamma_b}, *)$ be a graphical fuzzy b-metric space, $T : O \to O$ be a mapping and Γ' be a subgraph of Γ with $\Delta \subseteq \Sigma(\Gamma')$. Then, the mapping T is said to be a (Γ, Γ')-fuzzy graphical contraction on O whenever the criteria below are met:

(FGC1) Mapping T is edge preserving in $\Sigma(\Gamma')$, specifically, $(x, y) \in \Sigma(\Gamma')$ implies $(Tx, Ty) \in \Sigma(\Gamma')$;

(FGC2) There is $k \in \left(0, \frac{1}{b}\right)$ for which every pair $x, y \in O$ satisfying $(x, y) \in \Sigma(\Gamma')$,

$$M_{\Gamma_b}(Tx, Ty, kt) \geq M_{\Gamma_b}(x, y, t) \text{ for each } t > 0.$$

A sequence $\{x_n\}$ in O beginning with $x_0 \in O$ is called a T-Picard sequence (i.e., the Picard sequence associated with T) if $x_n = Tx_{n-1}$ is satisfied for every $n \in \mathbb{N}$. For the subsequent analysis, graph Γ' is assumed to be a subgraph of Γ in which $\Delta \subseteq \Sigma(\Gamma')$.

Theorem 1. *Suppose $(O, M_{\Gamma_b}, *)$ is a Γ'-complete graphical fuzzy b-metric space, and $T : O \to O$ is a (Γ, Γ')-fuzzy graphical contraction. Assume following conditions are satisfied:*

(i) there exists $x_0 \in O$ satisfying $Tx_0 \in [x_0]^l_{\Gamma'}$ for some $l \in \mathbb{N}$;

(ii) if a Γ'-termwise connected T-Picard sequence $\{x_n\}$ converges to some $x \in O$, then both limit $x \in O$ and $n_0 \in \mathbb{N}$ exist satisfying $(x_n, x) \in \Sigma(\Gamma')$ or $(x, x_n) \in \Sigma(\Gamma')$ at time t for all $n \geq n_0$;

(iii) $\lim_{t\to\infty} M_{\Gamma_b}(x, y, t) = 1$ for $x, y \in O$ satisfying $(x, y) \in \Sigma(\Gamma')$.

Consequently, there exists $\varpi^* \in O$ where the T-Picard sequence $\{x_n\}$ begins with $x_0 \in O$ is Γ'-termwise connected and converges to both ϖ^* and $T\varpi^*$ in O.

Proof. Suppose $x_0 \in O$ satisfies $Tx_0 \in [x_0]^l_{\Gamma'}$ for some $l \in \mathbb{N}$. Then, construct a T-Picard sequence $\{x_n\}$ with starting point x_0. It follows that there exists a path $\{y_i\}^l_{i=0}$ where $x_0 = y_0, Tx_0 = y_l$ and $(y_{i-1}, y_i) \in \Sigma(\Gamma')$ for $i = 1, 2, \ldots, l$. Given T as a (Γ, Γ')-fuzzy graphical contraction, the condition (FGC1) ensures that $(Ty_{i-1}, Ty_i) \in \Sigma(\Gamma')$ for $i = 1, 2, \ldots, l$. Consequently, $\{Ty_i\}^l_{i=0}$ forms a path from $Ty_0 = Tx_0 = x_1$ to $Ty_l = T^2x_0 = x_2$ of length l, which establishes that $x_2 \in [x_1]^l_{\Gamma'}$. Continue in similar manner, it can be derived that $(T^ny_{i-1}, T^ny_i) \in \Sigma(\Gamma')$ for $i = 1, 2, \ldots, l$. Accordingly, $\{T^ny_i\}^l_{i=0}$ is a path from $T^ny_0 = T^nx_0 = x_n$ to $T^ny_l = T^{n+1}x_0 = x_{n+1}$ of length l, which establishes that $x_{n+1} \in [x_n]^l_{\Gamma'}$ for any $n \in \mathbb{N}$. Therefore, $\{x_n\}$ is considered Γ'-termwise connected sequence.

As $(T^ny_{i-1}, T^ny_i) \in \Sigma(\Gamma')$ for $i = 1, 2, \ldots, l$ and $n \in \mathbb{N}$, by applying (FGC2), we yield

$$M_{\Gamma_b}(T^ny_{i-1}, T^ny_i, t) \geq M_{\Gamma_b}\left(T^{n-1}y_{i-1}, T^{n-1}y_i, \frac{t}{k}\right)$$

for all $t > 0$. Repeating the above argument yield

$$\begin{aligned} M_{\Gamma_b}(T^ny_{i-1}, T^ny_i, t) &\geq M_{\Gamma_b}\left(T^{n-1}y_{i-1}, T^{n-1}y_i, \frac{t}{k}\right) \\ &\geq M_{\Gamma_b}\left(T^{n-2}y_{i-1}, T^{n-2}y_i, \frac{t}{k^2}\right) \\ &\;\;\vdots \\ &\geq M_{\Gamma_b}\left(y_{i-1}, y_i, \frac{t}{k^n}\right) \end{aligned} \tag{1}$$

for every $t > 0$. Since sequence $\{x_n\}$ is a Γ'-termwise connected sequence, by applying (1) and (GF$_b$MS4), one obtains, for any $n \in \mathbb{N}$ and $t > 0$,

$$\begin{aligned} M_{\Gamma_b}(x_n, x_{n+1}, t) &= M_{\Gamma_b}(T^n y_0, T^n y_l, t) \\ &\geq M_{\Gamma_b}\left(T^n y_0, T^n y_1, \frac{t}{2b}\right) * M_{\Gamma_b}\left(T^n y_1, T^n y_l, \frac{t}{2b}\right) \\ &\geq M_{\Gamma_b}\left(T^n y_0, T^n y_1, \frac{t}{2b}\right) * M_{\Gamma_b}\left(T^n y_1, T^n y_2, \frac{t}{(2b)^2}\right) * M_{\Gamma_b}\left(T^n y_2, T^n y_l, \frac{t}{(2b)^2}\right) \\ &\vdots \\ &\geq M_{\Gamma_b}\left(T^n y_0, T^n y_1, \frac{t}{2b}\right) * M_{\Gamma_b}\left(T^n y_1, T^n y_2, \frac{t}{(2b)^2}\right) * \cdots * M_{\Gamma_b}\left(T^n y_{l-1}, T^n y_l, \frac{t}{(2b)^{l-1}}\right) \\ &\geq M_{\Gamma_b}\left(y_0, y_1, \frac{t}{k^n 2b}\right) * M_{\Gamma_b}\left(y_1, y_2, \frac{t}{k^n (2b)^2}\right) * \cdots * M_{\Gamma_b}\left(y_{l-1}, y_l, \frac{t}{k^n (2b)^{l-1}}\right). \end{aligned}$$

Furthermore, due to the fact that sequence $\{x_n\}$ is considered Γ'-termwise connected sequence, it follows that for $n, m \in \mathbb{N}$ satisfing $m > n$ and $t > 0$, we obtain

$$\begin{aligned} M_{\Gamma_b}(x_n, x_m, t) &\geq M_{\Gamma_b}\left(x_n, x_{n+1}, \frac{t}{2b}\right) * M_{\Gamma_b}\left(x_{n+1}, x_m, \frac{t}{2b}\right) \\ &\geq M_{\Gamma_b}\left(x_n, x_{n+1}, \frac{t}{2b}\right) * M_{\Gamma_b}\left(x_{n+1}, x_{n+2}, \frac{t}{(2b)^2}\right) * M_{\Gamma_b}\left(x_{n+2}, x_m, \frac{t}{(2b)^2}\right) \\ &\vdots \\ &\geq M_{\Gamma_b}\left(x_n, x_{n+1}, \frac{t}{2b}\right) * M_{\Gamma_b}\left(x_{n+1}, x_{n+2}, \frac{t}{(2b)^2}\right) * \cdots * M_{\Gamma_b}\left(x_{m-1}, x_m, \frac{t}{(2b)^{m-n-1}}\right) \\ &\geq \left[M_{\Gamma_b}\left(y_0, y_1, \frac{t}{k^n (2b)^2}\right) * M_{\Gamma_b}\left(y_1, y_2, \frac{t}{k^n (2b)^3}\right) * \cdots * M_{\Gamma_b}\left(y_{l-1}, y_l, \frac{t}{k^n (2b)^l}\right)\right] \\ &\quad * \left[M_{\Gamma_b}\left(y_0, y_1, \frac{t}{k^{n+1} (2b)^3}\right) * M_{\Gamma_b}\left(y_1, y_2, \frac{t}{k^{n+1} (2b)^4}\right) * \cdots * M_{\Gamma_b}\left(y_{l-1}, y_l, \frac{t}{k^{n+1} (2b)^{l+1}}\right)\right] * \cdots \\ &\quad * \left[M_{\Gamma_b}\left(y_0, y_1, \frac{t}{k^{m-1} (2b)^{m-n}}\right) * M_{\Gamma_b}\left(y_1, y_2, \frac{t}{k^{m-1} (2b)^{m-n+1}}\right) * \cdots \right. \\ &\quad \left. * M_{\Gamma_b}\left(y_{l-1}, y_l, \frac{t}{k^{m-1} (2b)^{l+m-n-2}}\right)\right]. \end{aligned}$$

Since $k \in \left(0, \frac{1}{b}\right)$, letting $n, m \to \infty$ and utilizing condition (iii), one deduces that

$$\lim_{n,m\to\infty} M_{\Gamma_b}(x_n, x_m, t) = [1 * 1 * \cdots * 1] * [1 * 1 * \cdots * 1] * \cdots * [1 * 1 * \cdots * 1] = 1.$$

Consequently, $\{x_n\}$ is a Γ'-termwise connected Cauchy sequence in O. Given that $(O, M_{\Gamma_b}, *)$ is Γ'-complete, the sequence $\{x_n\}$ converges to some $\varpi^* \in O$. Employing condition (ii), there exists $\varpi^* \in O$ and $n_0 \in \mathbb{N}$ satisfying $(x_n, \varpi^*) \in \Sigma(\Gamma')$ or $(\varpi^*, x_n) \in \Sigma(\Gamma')$ for $n > n_0$ and

$$\lim_{n\to\infty} M_{\Gamma_b}(x_n, \varpi^*, t) = 1 \text{ for all } t > 0.$$

Without the loss of generality, we consider $(x_n, \varpi^*) \in \Sigma(\Gamma')$ for all $n > n_0$, as the case $(\varpi^*, x_n) \in \Sigma(\Gamma')$ can be treated in a similar manner. By (FGC1), it follows that $(x_{n+1}, T\varpi^*) = (Tx_n, T\varpi^*) \in \Sigma(\Gamma')$ for all $n > n_0$. Now, applying (FGC2), we yield

$$M_{\Gamma_b}(x_{n+1}, T\varpi^*, t) = M_{\Gamma_b}(Tx_n, T\varpi^*, t) \geq M_{\Gamma_b}\left(x_n, \varpi^*, \frac{t}{k}\right), \text{ for all } n \geq n_0.$$

On account of $\lim_{n\to\infty} M_{\Gamma_b}(x_n, \varpi^*, t) = 1$ for every $t > 0$, taking $n \to \infty$ in the above inequality leads to

$$\lim_{n\to\infty} M_{\Gamma_b}(x_{n+1}, T\varpi^*, t) = 1.$$

Hence, $\{x_n\}$ converges to both ϖ^* and $T\varpi^*$ in O. □

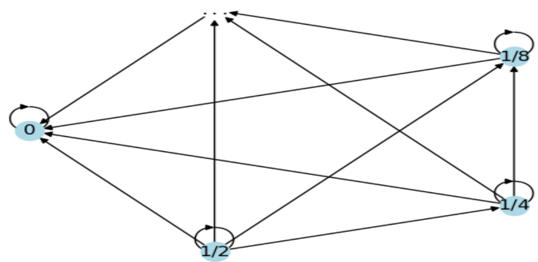

Figure 5: The graph structure corresponding to graphical fuzzy b-metric space.

Remark 3. The preceding finding establishes only that the T-Picard sequence constructed via a (Γ, Γ')-fuzzy graphical contraction on a Γ'-complete graphical fuzzy b-metric space is convergent. However, this finding does not guarantee the presence of a fixed point of T, as illustrated below.

Example 5. Suppose $O = \left\{\frac{1}{2^n} : n \in \mathbb{N}\right\} \cup \{0\}$ and Γ is a graph defined by $\Xi(\Gamma) = O$ and $\Sigma(\Gamma) = \Delta \cup \{(x, y) \in O \times O : y \leq x\}$ (see Figure 5). Define a fuzzy set $M_{\Gamma_b} : O \times O \times (0, \infty) \to [0, 1]$ by

$$M_{\Gamma_b}(x, y, t) = \frac{t}{t + |x - y|^2}, \text{ for all } x, y \in O \text{ and } t > 0.$$

Accordingly, $(O, M_{\Gamma_b}, *_m)$ forms a Γ'-complete graphical fuzzy b-metric space with $b = 2$. Define a mapping $T : O \times O$ expressed by

$$T(x) = \begin{cases} \frac{x}{2}, & x \in O \setminus \{0\}; \\ \frac{1}{2}, & x = 0. \end{cases}$$

Let Γ' be a subgraph of Γ with $\Xi(\Gamma') = O$ and $\Sigma(\Gamma') = \Sigma(\Gamma)$. Accordingly, the mapping T is readily seen to be a (Γ, Γ')-fuzzy graphical contraction with some $k < \frac{1}{2}$. Furthermore, for any $x, y \in O$ constrained by $(x, y) \in \Sigma(\Gamma')$, the condition $\lim_{t\to\infty} M_{\Gamma_b}(x, y, t) = 1$ holds. Now, pick an arbitrary $x_0 \in O \setminus \{0\}$. Then, we have $(x_0, Tx_0) \in \Sigma(\Gamma')$, which implies that $Tx_0 \in [x_0]^l_{\Gamma'}$ for $l = 1$. Consider any T-Picard sequence $\{x_n\}$ begins with x_0. Then, each term of the sequence takes the form $x_n = \frac{x_0}{2^{n+1}}$ for all $n \in \mathbb{N}$. In addition, this sequence $\{x_n\}$ is Γ'-termwise connected and converges to every $\varpi^* \in O$. For this particular choice of x_0, it can also be observed that for each $\varpi^* \in O$, there is $n_0 \in \mathbb{N}$ such that either $(x_n, \varpi^*) \in \Sigma(\Gamma')$ or $(\varpi^*, x_n) \in \Sigma(\Gamma')$ at time t for all $n \geq n_0$. Consequently, every criterion of Theorem 1 is fulfilled. Nonetheless, mapping T possess no fixed point in O.

With the aid of the property defined below, the next result establishes the fixed point appearance.

Definition 13. Suppose $(O, M_{\Gamma_b}, *)$ is a graphical fuzzy b-metric space, with Γ' a subgraph of Γ, and $T : O \to O$ is a self-mapping. Accordingly, $(O, M_{\Gamma_b}, *, \Gamma', T)$ possess property (W), if any Γ'-termwise connected T-Picard sequence $\{x_n\}$ admits two limits $\varpi^* \in O$ and $\vartheta^* \in TO$, then $\varpi^* = \vartheta^*$.

Theorem 2. *Suppose $(O, M_{\Gamma_b}, *)$ is a Γ'-complete graphical fuzzy b-metric space and $T : O \to O$ is a (Γ, Γ')-fuzzy graphical contraction. Assume following conditions are satisfied:*

(i) there exists $x_0 \in O$ satisfying $Tx_0 \in [x_0]^l_{\Gamma'}$, for some $l \in \mathbb{N}$;

(ii) if a Γ'-termwise connected T-Picard sequence $\{x_n\}$ converges to some $x \in O$, then both limit $x \in O$ and $n_0 \in \mathbb{N}$ exist satisfying $(x_n, x) \in \Sigma(\Gamma')$ or $(x, x_n) \in \Sigma(\Gamma')$ at time t for all $n \geq n_0$.

(iii) $\lim_{t\to\infty} M_{\Gamma_b}(x, y, t) = 1$ for $x, y \in O$ satisfying $(x, y) \in \Sigma(\Gamma')$.

Consequently, there exists $\varpi^* \in O$ where the T-Picard sequence $\{x_n\}$ begins with $x_0 \in O$ is Γ'-termwise connected and converges to both ϖ^* and $T\varpi^*$ in O. In addition, if $(O, M_{\Gamma_b}, *, \Gamma', T)$ possess property (W), then T possess at least one fixed point in O.

Proof. It can be concluded from Theorem 1 that the T-Picard sequence $\{x_n\}$ begins with value $x_0 \in O$ converges to both ϖ^* and $T\varpi^*$ in O. Clearly, $\varpi^* \in O$ and $T\varpi^* \in TO$. Consequently, by applying property (W), it leads to $\varpi^* = T\varpi^*$ which verifies ϖ^* is fixed under T. □

In Theorem 2, property (W) is not superfluous. This is confirmed by Example 5, which demonstrates that the absence of this condition results in the failure of fixed point existence, although the remaining conditions are satisfied. Also, it is important to note that the preceding theorem establishes the presence of a fixed point for mapping T, but does not guarantee its uniqueness. This is further clarified by the subsequent instance.

Example 6. Suppose $O = [0,1]$ and Γ is a graph with $\Xi(\Gamma) = O$ and $\Sigma(\Gamma) = \Delta \cup \{(x,y) \in O \times O : x \leq y\}$. Define a fuzzy set $M_{\Gamma_b} : O \times O \times (0,\infty) \to [0,1]$ by

$$M_{\Gamma_b}(x,y,t) = \frac{t}{t + |x-y|^2}, \text{ for all } x,y \in O \text{ and } t > 0.$$

Accordingly, $(O, M_{\Gamma_b}, *_m)$ constitutes Γ'-complete graphical fuzzy b-metric space with $b = 2$. Consider mapping $T : O \times O$ defined by

$$T(x) = \begin{cases} 1, & x = 1; \\ \frac{x}{2}, & \text{otherwise.} \end{cases}$$

Let Γ' be a subgraph of Γ with $\Xi(\Gamma') = O$ and $\Sigma(\Gamma') = \Sigma(\Gamma)$. Accordingly, the mapping T is readily seen as a (Γ,Γ')-fuzzy graphical contraction with some $k < \frac{1}{2}$. Furthermore, for any $x,y \in O$ constrained by $(x,y) \in \Sigma(\Gamma')$, condition $\lim_{t\to\infty} M_{\Gamma_b}(x,y,t) = 1$ holds. Now, pick an arbitrary $x_0 \in O \setminus \{1\}$. Then, one obtains $(x_0, Tx_0) \in \Sigma(\Gamma')$, which implies that $Tx_0 \in [x_0]^l_{\Gamma'}$ for $l = 1$. Consider any T-Picard sequence $\{x_n\}$ begins with x_0. Then, each term of the sequence takes the form $x_n = \frac{x_0}{2^n}$ for all $n \in \mathbb{N}$. In addition, this sequence $\{x_n\}$ is Γ'-termwise connected and converges to every $\varpi^* \in O$. For this particular choice of x_0, it can also be observed that for each $\varpi^* \in O$, there exists $n_0 \in \mathbb{N}$ satisfying either $(x_n, \varpi^*) \in \Sigma(\Gamma')$ or $(\varpi^*, x_n) \in \Sigma(\Gamma')$ at time t for each $n \geq n_0$. It can be observed that $(O, M_{\Gamma_b}, *, \Gamma', T)$ satisfies property (W). Consequently, each conditions of Theorem 2 is met. In particular, 0 and 1 are fixed under mapping T in O.

The collection of fixed points of a mapping T is expressed as $\text{Fix}(T)$. Furthermore, the notation O_T is defined by $O_T = \{x \in O : (x, Tx) \in \Sigma(\Gamma')\}$. The result below provides a necessary condition for the uniqueness of a fixed point.

Theorem 3. *Suppose $(O, M_{\Gamma_b}, *)$ is a Γ'-complete graphical fuzzy b-metric space and $T : O \to O$ is a (Γ,Γ')-fuzzy graphical contraction. Assuming Theorem 2 applies and O_T, as a subgraph of Γ', is connected, then mapping T admits a unique fixed point.*

Proof. Theorem 2 ensures the existence of fixed point of T. Assume that O_T is connected subgraph of Γ', and that ϖ^*, ϑ^* represents two separate fixed points of T. The fact that $\Delta \subseteq \Sigma(\Gamma')$ validates $\text{Fix}(T) \subseteq O_T$ and hence, $\varpi^*, \vartheta^* \in O_T$. Since O_T is connected, we deduce $(\varpi^* P \vartheta^*)_\Gamma$. Consequently, a sequence $\{x_i\}_{i=0}^l$ exists such that $x_0 = \varpi^*, x_l = \vartheta^*$ and $(x_{i-1}, x_i) \in \Sigma(\Gamma')$ for $i = 1,2,\ldots,l$.

Given that T is a (Γ,Γ')-graphical fuzzy contraction, by applying (FGC1), we yield $(T^n x_{i-1}, T^n x_i) \in \Sigma(\Gamma')$ for $i = 1,2,\ldots,l$ and for each $n \in \mathbb{N}$. Henceforth, by applying (FGC2), this yields

$$\begin{aligned} M_{\Gamma_b}(T^n x_{i-1}, T^n x_i, t) &\geq M_{\Gamma_b}\left(T^{n-1}x_{i-1}, T^{n-1}x_i, \frac{t}{k}\right) \\ &\geq M_{\Gamma_b}\left(T^{n-2}x_{i-1}, T^{n-2}x_i, \frac{t}{k^2}\right) \\ &\ \ \vdots \\ &\geq M_{\Gamma_b}\left(x_{i-1}, x_i, \frac{t}{k^n}\right) \end{aligned}$$

for $i = 1,2,\ldots,l$ and for each $n \in \mathbb{N}$. By applying (GF$_b$MS4), it follows that

$$\begin{aligned} M_{\Gamma_b}(T^n\varpi^*, T^n\vartheta^*, t) &= M_{\Gamma_b}(T^n x_0, T^n x_l, t) \\ &\geq M_{\Gamma_b}\left(T^n x_0, T^n x_1, \frac{t}{2b}\right) * M_{\Gamma_b}\left(T^n x_1, T^n x_l, \frac{t}{2b}\right) \\ &\geq M_{\Gamma_b}\left(T^n x_0, T^n x_1, \frac{t}{2b}\right) * M_{\Gamma_b}\left(T^n x_1, T^n x_2, \frac{t}{(2b)^2}\right) * M_{\Gamma_b}\left(T^n x_2, T^n x_l, \frac{t}{(2b)^2}\right) \\ &\ \ \vdots \\ &\geq M_{\Gamma_b}\left(T^n x_0, T^n x_1, \frac{t}{2b}\right) * M_{\Gamma_b}\left(T^n x_1, T^n x_2, \frac{t}{(2b)^2}\right) * \cdots * M_{\Gamma_b}\left(T^n x_{l-1}, T^n x_l, \frac{t}{(2b)^{l-1}}\right) \\ &\geq M_{\Gamma_b}\left(x_0, x_1, \frac{t}{k^n 2b}\right) * M_{\Gamma_b}\left(x_1, x_2, \frac{t}{k^n(2b)^2}\right) * \cdots * M_{\Gamma_b}\left(x_{l-1}, x_l, \frac{t}{k^n(2b)^{l-1}}\right). \end{aligned}$$

The fact that $\varpi^*, \vartheta^* \in \text{Fix}(T)$ indicates $T^n\varpi^* = \varpi^*$ and $T^n\vartheta^* = \vartheta^*$. Since $k \in \left(0, \frac{1}{b}\right)$, for the preceding inequality, by letting $n \to \infty$ and along with the condition (iii), it follows that

$$M_{\Gamma_b}(\varpi^*, \vartheta^*, t) = 1 \text{ for all } t > 0.$$

Equivalently, $\varpi^* = \vartheta^*$, leading to a contradiction with the initial assumption. Hence, T admits a unique fixed point. □

Remark 4. Referring to the existing literature discussed in the previous sections, specifically within the graphical metric structures, weakly connectivity of O_T is often imposed to secure the uniqueness of a fixed point. In developing the graphical fuzzy metric setting, Saleem et al. [18] explored conditions for the uniqueness of a fixed point using similar reasoning. However, Shukla et al. [22] reported a flaw in this approach, as weakly connectivity permits undirected paths, which may result in the failure of uniqueness. To address this matter, they refined the condition by assuming O_T is connected. Our analysis adopts the approach proposed by Shukla et al.

Remark 5. As discussed in the previous section, graphical fuzzy b-metric spaces can be interpreted as a natural extension of graphical fuzzy metric spaces. By setting $b = 1$ in Theorem 1, Theorem 2 and Theorem 3, the results presented in [18] and [22] can be deduced as corollaries.

This section concludes with the existing contractive results in fuzzy metric setting being derived as corollaries of the preceding result.

Corollary 1. *Let $(O, M_b, *)$ be a complete fuzzy b-metric space and $\lim_{t\to\infty} M_b(x, y, t) = 1$ for $x, y \in O$. Suppose that $T : O \to O$ is a mapping satisfying $M_b(Tx, Ty, kt) \geq M_b(x, y, t)$ for all $x, y \in O$ and $t > 0$, with $k \in \left(0, \frac{1}{b}\right)$. Then, mapping T admits a unique fixed point.*

Proof. Following Remark 2, let graphs Γ and Γ' be defined by $\Xi(\Gamma) = \Xi(\Gamma') = O$ and $\Sigma(\Gamma) = \Sigma(\Gamma') = O \times O$. This construction yields a graphical fuzzy b-metric space. Accordingly, the criteria of Theorem 3 hold true, and the result follows. □

Corollary 2 ([13])**.** *Let $(O, M, *)$ be a complete fuzzy metric space and $\lim_{t\to\infty} M(x, y, t) = 1$ for $x, y \in O$. Suppose that $T : O \to O$ is a mapping satisfying $M(Tx, Ty, kt) \geq M(x, y, t)$ for all $x, y \in O$ and $t > 0$, with $k \in (0, 1)$. Then, mapping T admits a unique fixed point.*

Proof. By considering graphs Γ and Γ' where $\Xi(\Gamma) = \Xi(\Gamma') = O$ together with $\Sigma(\Gamma) = \Sigma(\Gamma') = O \times O$, and setting $b = 1$, this result is deduced directly from Theorem 3. □

5 Application to the solution of equations of motion

This section illustrates the applicability of the main result in verifying the solvability of the equations of motion. Equations of motion represent a mathematical framework to illustrate the connection between the object's position, velocity, acceleration, and time. These equations are critical in engineering, particularly in the analysis of the dynamics of mechanical systems. A multibody mechanical system refers to a product constructed from mechanical, electrical or other interconnected components that are permitted to move relative to one another. Such systems may range in complexity from simple mechanisms like pendulums and slide-crank mechanisms, to sophisticated designs such as automotive suspension and steering systems, or walking robotic devices. The analysis of equations of motion in these mechanical systems enables the engineering profession to create and deliver high quality, marketable products, while also optimize both the design and operational performance of these systems.

Beyond engineering, the equations of motion also play a significant role in geophysical fluid dynamics, which concerns the study of fluid motion in natural systems, such as the Earth's atmosphere and oceans. These equations describe how the fluids flow and evolve under the influence of external forces or thermal variations, thereby addressing both dynamic and thermodynamic aspects of the system. It is worth noting that the equations of motion governing fluid mechanics are conceptually distinct from those used in rigid-body mechanics. In rigid-body problems, the analysis typically focuses on determining values such as velocity and density at fixed points in space as time progresses. In contrast, due to the continuous and deformable nature of fluids, the analysis involves examining how the entire fields of dynamical variables evolve in both space and time. This approach is commonly referred to as Eulerian viewpoint. Further details and discussions on these topics can be found in [23, 17].

Inspired by the works in [16], [25], and [24], we consider a simple problem: A body with mass m is initially stationary, that is, $x = 0$ at time $r = 0$. A force f is applied to the body, causing it to move in the horizontal direction, with its velocity increasing from 0 to 1 immediately after $r = 0$. The objective is to identify the position of the body at a given time r. The body's dynamics are formulated by the second-order differential equation that follows:

$$\begin{cases} m\frac{d^2x}{dr^2} = f(r, x(r)), \\ x(0) = 0, x'(0) = 1, \end{cases} \tag{2}$$

for every $r \in [0,1]$, where real-valued function f is defined over the set $[0,1] \times \mathbb{R}$. Solving (2) is equivalent to finding the solution for the subsequent integral equation

$$x(r) = \int_0^1 G(r,s) f(s, x(s)) ds, \quad r \in [0,1],$$

in which $G(r,s)$ denotes the associated Green's function, expressed as

$$G(r,s) = \begin{cases} r, & r \leq s; \\ 2r - s, & r \geq s. \end{cases}$$

Define a mapping $S = C([0,1], \mathbb{R}) \times C([0,1], \mathbb{R})$ by

$$Sx(r) = \int_0^1 G(r,s) f(s, x(s)) ds, \quad x \in C([0,1], \mathbb{R}), r \in [0,1].$$

In what follows, we demonstrate that if a fixed point of mapping S exists, the integral equation is solvable, which in turn implies the solvability of the above problem.

Suppose $O = C([0,1], \mathbb{R})$ and Γ is a graph defined by $\Xi(\Gamma) = O$ and $\Sigma(\Gamma) = \Delta \cup \{(x,y) \in O \times O : x(r) \leq y(r) \text{ for all } r \in [0,1]\}$. Fuzzy set $M_{\Gamma_b} : O \times O \times (0, \infty) \to [0,1]$ expressed by

$$M_{\Gamma_b}(x,y,t) = \exp\left\{ -\frac{\sup_{r\in[0,1]} |x(r) - y(r)|^2}{t} \right\}, \text{ for all } x, y \in O \text{ and } t > 0.$$

In additon, consider $\Gamma' = \Gamma$. Then, $(O, M_{\Gamma_b}, *_m)$ forms Γ'-complete graphical fuzzy b-metric space with $b = 2$.

Theorem 4. *Suppose $(O, M_{\Gamma_b}, *_m)$ is a Γ'-complete graphical fuzzy b-metric space defined as above. Consider an integral operator $S : O \to O$ given as:*

$$Sx(r) = \int_0^1 G(r,s) f(s, x(s)) ds$$

for all $x \in O$ and $r \in [0,1]$. Conditions below are assumed to hold:

(i) For all $x, y \in O$ and $s \in [0,1]$,

$$|f(s, x(s)) - f(s, y(s))| \leq |x(s) - y(s)|;$$

(ii) For all $r, s \in [0,1]$ and some $k \in \left(0, \frac{1}{b}\right)$,

$$\sup_{r\in[0,1]} \int_0^1 G(r,s) ds \leq \sqrt{k}.$$

Then, the (2) has a solution in O.

Proof. Observe that the operator S is well-defined. Now, for any $x, y \in O$ satisfying $(x,y) \in \Sigma(\Gamma')$ and $r \in [0,1]$, we have

$$\begin{aligned} |Sx(r) - Sy(r)| &= \left| \int_0^1 G(r,s) f(s, x(s)) ds - \int_0^1 G(r,s) f(s, y(s)) ds \right| \\ &= \left| \int_0^1 G(r,s) \{ f(s, x(s)) - f(s, y(s)) \} ds \right| \\ &\leq \int_0^1 G(r,s) \left| f(s, x(s)) - f(s, y(s)) \right| ds \\ &\leq \int_0^1 G(r,s) |x(s) - y(s)| ds. \end{aligned}$$

Therefore, for all $t > 0$, it follows that

$$\begin{aligned} M_{\Gamma_b}(Sx(r), Sy(r), kt) &= \exp\left\{-\frac{\sup_{r\in[0,1]} |Sx(r) - Sy(r)|^2}{kt}\right\} \\ &= \exp\left\{-\frac{\sup_{r\in[0,1]} \left(\int_0^1 G(r,s)|x(s) - y(s)|ds\right)^2}{kt}\right\} \\ &\geq \exp\left\{-\frac{|x(s) - y(s)|^2 \left(\sup_{r\in[0,1]} \int_0^1 G(r,s)ds\right)^2}{kt}\right\} \\ &\geq \exp\left\{-\frac{|x(s) - y(s)|^2 \left(\sqrt{k}\right)^2}{kt}\right\} \\ &= \exp\left\{-\frac{|x(s) - y(s)|^2}{t}\right\} \\ &\geq \exp\left\{-\frac{\sup_{r\in[0,1]} |x(r) - y(r)|^2}{t}\right\} \\ &= M_{\Gamma_b}(x(r), y(r), t). \end{aligned}$$

Consequently, mapping S constitutes a (Γ, Γ')-fuzzy graphical contraction. The rest of the assumptions in Theorem 2 can be verified accordingly. Therefore, a fixed point $\varpi^* \in O$ of S is established, which indicates the existence of solution for differential equation (2). □

6 Conclusion

The introduction of graphical structures into fuzzy metric frameworks marks a notable breakthrough in the development of fixed point theory. This interplay strengthens both theories and enhances their applicability in real-world problems. The present study introduced graphical fuzzy b-metric spaces framework. Several topological aspects of the proposed spaces are defined, and several fixed point outcomes are obtained. To support our theoretical findings, instances and an application to equations of motion are additionally provided. Despite these advancements, this line of research remains largely unexplored and holds considerable potential for further investigation. To conclude this paper, we suggest the following open problems for future research:

(a) Investigate the fixed point results under diverse existing contractive constraints within graphical fuzzy metric structures.

(b) Extend the underlying abstract spaces using the concept of hypergraphs and discuss the corresponding fixed point results.

(c) Explore best proximity point theory or fixed figure problems under the paradigm of graphical fuzzy metric spaces.

Declarations

Ethics approval and consent to participate

Not applicable.

Consent for publication

Not applicable.

Availability of data and materials

No data were used to support this study.

Competing interests

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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Authors' contributions

Koon Sang Wong, Zabidin Salleh, Mudasir Younis and Utkir Nematovich Kuljanov were responsible for material preparation, data collection, and analysis. The study was conceptualized and designed collaboratively by all authors. Koon Sang Wong prepared the initial draft of the manuscript, and all authors contributed feedback to subsequent revisions. All authors have read and approved the final manuscript.

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